Math 16A Discussion Problem Set Solutions
RRR Week

December 25th

I selected problems from many places in the textbook (and from finals I found online), with more emphasize on the calculus portion. There are sections missing so this will not be a comprehensive review of all of 16a.

1 Integrals

Compute the following integrals

1. $\int \frac{\sqrt{t}}{2} dt$
2. $\int \frac{-u}{2-u^2} du$
3. $\int_0^1 x\sqrt{5x^2+4}dx$

Solutions

1. $(1/3)t^{3/2} + C$
2. Here we make the substitution $t = 2 - u^2$, then we should get: $(1/2)\ln|2-u^2| + C$
3. Here we make the substitution $u = 5x^2 + 4$, then we get$(5x^2 + 4x)^{3/2}(1/15) + C$

2 Implicit Differentiation

1. Compute dy/dx for $x^2y^3 + 4xy = 2$
2. Compute dy/dx for $\ln(x + y) = 1 + x^2 + y^3$.
3. What is the rate the volume of a sphere changes if its radius is 2 meters and is growing at 3 meters per second

Solutions These are all the same, take the derivative with respect to the independent variable, solve for the derivative.

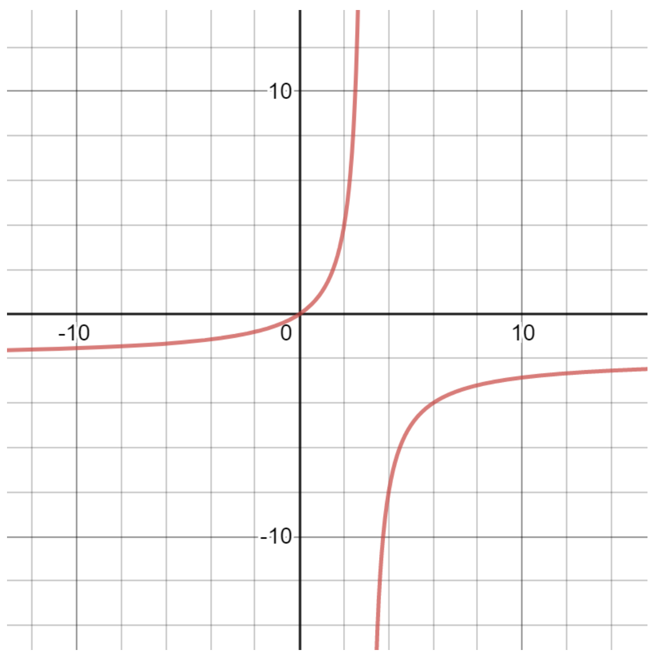
1. $2xy^3 + 3x^2y^2\frac{dy}{dx} + 4y + 4x\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = (-4y - 2x^3)/(3x^2 + y^2 + 4x)$
2. $\frac{y'}{x+y} = 1 + 3x + 3y^2y'$, so $y' = (1 + 3x)/((x + y)^{-1} + 3y^2)$.
3. $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dt} = \pi r^2\frac{dr}{dt}$, plug everything in to get $\pi \cdot 4 \cdot 3 = 12\pi$ meters per second.

3 Graphing

1. Graph $\frac{2x}{3-x}$ and $\ln(x^2 + 4)$.

Solutions

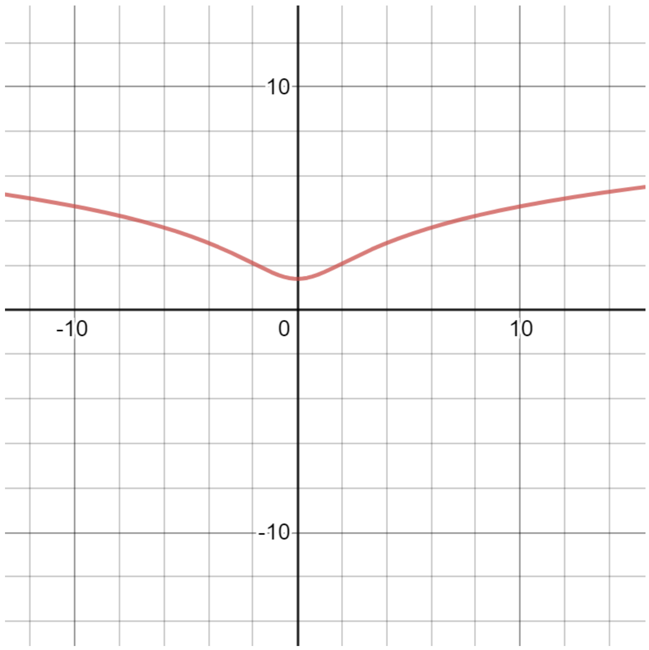
1. This is $f(x) = 1/x$ but we have shifted everything to the right 3 units and scaled everything by 2. So the graph should look like



for $f(x) = \ln(x^2 + 4)$, a little more needs to be done. The argument of ln must be > 0, therefore we require $x^2 + 4 > 0$, but this is always true (one way to see this is to plot $y = x^2 + 4$ and see that the graph is always above the $x-$axis. Therefore the domain is all x.

Note that $f(-x) = \ln((-x)^2+4) = \ln(x^2+4)$, therefore this function is even. $f(0) = \ln 4$. As $x \to \infty$, we see that $f(x) \to +\infty$. So it is reasonable to guess the function looks a little like $x^2 + \ln(4)$.

We should find the critical points, $f'(x) = \frac{2x}{|x^2+4|}$, this is zero only when $x = 0$, so that is the only critical point. Furthermore, $f''(x) = \frac{4x-4x^2}{(x^2+4)^2}$, this is zero if $x = -1, 0, 1$, plugging in points (or just thinking about $4x - 4x^2$) we see that f is concave down on $(-\infty, -1)$ and $(1, \infty)$ and concave up otherwise. So this is **different** from $x^2 + \ln(4)$, a plot looks like:



4 Derivatives

compute the following derivatives

1. $y = 5xe^{2x}$
2. $y = \ln(2 + x^2)$
3. $y = x^x$
4. $y = \log(1 + e^x)$

Solutions

1. $y' = 5e^{2x} + 10xe^{2x}$
2. $y' = \frac{2x}{|2+x^2|}$
3. $\ln(y) = x\ln(x)$, so $\frac{y'}{y} = \ln x + \frac{x}{x} = \ln x + 1$, so $y' = x^x(1+\ln(x))$
4. $y' = (\frac{1}{\ln(10)})\frac{e^x}{1+e^x}$
5. Determine the area of the region enclosed by the three curves: $f(x) = \frac{-x}{2} + \frac{3\pi}{8} + \frac{\sqrt{2}}{2}$, $g(x) = x^2 + x - 1 - \frac{3\pi}{2} - (\frac{3\pi}{2})^2$ and $h(x) = \sin(x)$

Misc

1. Compute $\lim_{x\to 0} \frac{\ln\cos 5x}{x^2}$
2. $\int 1 + e^x + 2\sin(x)dx$
3. find the 43 derivative of $\sin(2x)$
4. compute the second derivative of $\sin(x)\cos(x)$
5. Compute $\lim_{x\to\infty}(1+\frac{2}{x})^x$
6. Compute $\frac{d}{dx}\int_0^{\sin(x)}(x - e^{t^2})dt$

Solutions

1. Using L'Hopital's rule, the derivatives of the numerator and denominator are (respectively) $\frac{-5\sin(5x)}{5\cos(5x)}$ and $2x$, so our new fraction we need to compute the limit for is $\frac{-5\sin(5x)}{10x\cos(5x)}$. Let's do this again, to get:

$$\lim_{x\to 0}\frac{-25\cos(5x)}{10\cos(5x) - 50x\sin(x)} = \frac{-25}{10} = -2.5 \qquad (1)$$

2. $x + e^x - 2\cos(x) + C$
3. let $y = \sin(2x)$, so $y' = 2\cos(2x)$, $y'' = -4\sin(2x)$, $y''' = -2^3\cos(2x)$, $y'''' = 2^4\sin(2x)$. Therefore $y^{(4k)} = 2^{4k}\sin(2x)$ for k a positive integer. So $y^{(43)} = \frac{d^3}{dx^3}(2^{40}\sin(2x)) = -2^{43}\cos(2x)$.
4. Let $y = \sin(x)\cos(x)$, then $y' = \cos(x)^2 - \sin(x)^2 = \cos(2x)$, so $y'' = -2\sin(2x) = -4\sin(x)\cos(x)$ (that was admittedly an odd way to do this problem, but the obvious way is boring).

5. Let $y = (1 + \frac{2}{x})^x$, so $\ln(y) = x\ln(1 + \frac{2}{x})$. Let's compute the limit of the right hand side as $x \to \infty$, to get:

$$\lim_{x\to\infty} \frac{\ln(1+\frac{2}{x})}{1/x} = \lim_{x\to\infty} \frac{\frac{1}{1+2/x}(-2/x^2)}{-1/x^2} = \lim_{x\to\infty} \frac{2}{1+2/x} = 2 \tag{2}$$

therefore $\lim_{x\to\infty} \ln y = 2$, and so $\lim_{x\to\infty} y = e^2$.

6. First we can write $\int_0^{\sin(x)} x - e^{t^2}\,dt = \sin(x)x - \int_0^{\sin(x)} e^{t^2}\,dt$. Let $F(t)$ be the antiderivative of e^{t^2}, therefore the original integral is just :

$$\sin(x)x - F(\sin(x)) + F(0) \tag{3}$$

so if we take derivatives is x, we get:

$$\cos(x)x + \sin(x) - F'(\sin(x))\cos(x) \tag{4}$$

but recall that $F'(x) = e^{x^2}$, therefore this just becomes: $\cos(x)x + \sin(x) - e^{\sin(x)^2}\cos(x)$

7. It is important to first graph what the region looks like. If you graph this, you will realize you need to compute three intersection points and compute two integrals. The first integral between $f(x)$ and $h(x)$, the second between $f(x)$ and $g(x)$. The intersection points can be computed with algebra as $x_1 = 3\pi/4$, $x_2 = 3\pi/2$ and (something very nasty) $x_3 = \frac{1}{4}(-3 + \sqrt{25 + 8\sqrt{2} + 30\pi + 36\pi^2})$. So then the area we would like is:

$$\int_{x_1}^{x_2} f(x) - g(x)dx + \int_{x_2}^{x_3} f(x) - h(x)dx \tag{5}$$

then you just compute these integrals.